

# Precision calculations in $\text{BR}(\bar{B} \rightarrow X_s \gamma)$ \*

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We briefly summarize the current status of perturbative calculations at next-to-next-to-leading-order (NNLO) accuracy in the  $\bar{B} \rightarrow X_s \gamma$  decay rate as well as that of non-perturbative power-corrections.

## 1. Introduction

Corrections to the  $\bar{B} \rightarrow X_s \gamma$  decay are usually described in the framework of an effective theory,<sup>1</sup>

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{QCD} \times \text{QED}}(u, d, s, c, b) + \frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^8 C_i^{\text{eff}}(\mu) O_i. \quad (1)$$

Here,  $C_i^{\text{eff}}$  are renormalization scale dependent effective couplings, the so-called Wilson coefficients, which encode the heavy gauge boson and the heavy top quark effects. The  $b$ -quark scale contributions, on the other hand, are seen as matrix elements of flavor changing operators,

$$\begin{aligned} O_1 &= (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L), & O_2 &= (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L), \\ O_{3,5} &= (\bar{s}_L \Gamma_{3,5} b_L) \sum_q (\bar{q} \Gamma'_{3,5} q), & O_{4,6} &= (\bar{s}_L \Gamma_{3,5} T^a b_L) \sum_q (\bar{q} \Gamma'_{3,5} T^a q), \\ O_7 &= \frac{\alpha_{\text{em}}}{4\pi} m_b (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}, & O_8 &= \frac{\alpha_s}{4\pi} m_b (\bar{s}_L \sigma^{\mu\nu} T^a b_R) G_{\mu\nu}^a, \end{aligned} \quad (2)$$

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<sup>1</sup> In writing (1) we discarded terms proportional to  $V_{ub} V_{us}^*$  since they give only small contributions to the branching ratio that start at next-to-leading-order (NLO). Similar NNLO corrections can therefore be safely neglected.

where  $\Gamma_3 = \gamma_\mu$ ,  $\Gamma'_3 = \gamma^\mu$ ,  $\Gamma_5 = \gamma_\mu \gamma_\nu \gamma_\lambda$  and  $\Gamma'_5 = \gamma^\mu \gamma^\nu \gamma^\lambda$ . Using (1), the differential decay rate for  $\bar{B} \rightarrow X_s \gamma$  can be written as follows,

$$d\Gamma = \frac{G_F^2 \alpha_{\text{em}} m_b^2}{256 \pi^6 m_B} |V_{tb} V_{ts}^*|^2 \frac{d^3 q}{E_\gamma} \sum_{i,j} C_i^{\text{eff}}(\mu) C_j^{\text{eff}}(\mu) W_{ij}(\mu). \quad (3)$$

In the equation displayed above,  $q$  denotes the momentum of the photon and the  $W_{ij}$ , which describe the hadronic dynamics, are given by the imaginary part of forward scattering amplitudes,

$$W_{ij}(\mu) = 2 \text{Im} \left( i \int d^4 x e^{-iq \cdot x} \langle \bar{B} | T \{ O_i^\dagger(x) O_j(0) \} | \bar{B} \rangle \right). \quad (4)$$

Since the mass of the  $b$ -quark is much larger than the binding energy of the  $B$ -meson, which is of the order of  $\Lambda \equiv \Lambda_{\text{QCD}}$ , we can perform an operator product expansion (OPE) of the time ordered products. Doing so, one finds that the leading term is the partonic decay rate which gives the dominant contribution, while the non-leading terms, the so-called power-corrections, are suppressed by powers of  $\Lambda/m_b$  and give non-vanishing contributions starting from  $O(\Lambda^2/m_b^2)$ .<sup>2</sup> In what follows we describe the state-of-the-art of perturbative and non-perturbative corrections in the  $\bar{B} \rightarrow X_s \gamma$  decay.

## 2. Perturbative corrections

The calculation of the perturbative corrections can be divided into three steps. In the first step one has to evaluate the effective couplings  $C_i^{\text{eff}}$  at the high-energy scale  $\mu \sim M_W$  by requiring equality of the Standard Model and the effective theory Green functions. Defining  $\tilde{\alpha}_s(\mu) = \alpha_s(\mu)/(4\pi)$ , the effective couplings can be expanded as follows,

$$C_i^{\text{eff}}(\mu) = C_i^{(0)\text{eff}}(\mu) + \tilde{\alpha}_s(\mu) C_i^{(1)\text{eff}}(\mu) + \tilde{\alpha}_s^2(\mu) C_i^{(2)\text{eff}}(\mu) + \dots \quad (5)$$

At NNLO accuracy one has to determine the coefficients  $C_i^{(2)\text{eff}}(\mu)$ . For  $i=7,8$  it required performing a three-loop calculation [1] whereas for the remaining cases  $i=1, \dots, 6$  a two-loop calculation was sufficient [2].

The second step involves the calculation of the anomalous dimension matrix  $\gamma^{\text{eff}}$  which describes the mixing of the operators under renormalization. Its knowledge is necessary to solve the effective theory renormalization

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<sup>2</sup> Strictly speaking, equation (4) and its OPE hold only for  $W_{77}$ . In all other cases the  $W_{ij}$  contain contributions in which the photon couples to light quarks ( $u, d, s, c$ ), and this leads to non-perturbative effects different from that mentioned above (see section 3). Furthermore, for  $W_{ij} \neq W_{77}$ , contributions that contain no photon in the final state should be excluded, as required by the experimental setup.

group equations for the effective couplings,

$$\mu \frac{d}{d\mu} C_i^{\text{eff}}(\mu) = \sum_j \gamma_{ji}^{\text{eff}} C_j^{\text{eff}}(\mu), \quad (6)$$

and to evolve the latter down to the low-energy scale  $\mu \sim m_b$ . Performing a perturbative expansion in the strong coupling constant, the anomalous dimension matrix takes the following form,

$$\gamma^{\text{eff}} = \tilde{\alpha}_s(\mu) \gamma^{(0)\text{eff}} + \tilde{\alpha}_s^2(\mu) \gamma^{(1)\text{eff}} + \tilde{\alpha}_s^3(\mu) \gamma^{(2)\text{eff}} + \dots \quad (7)$$

At NNLO one has to determine  $\gamma^{(2)\text{eff}}$  which is actually a  $8 \times 8$  matrix,

$$\gamma^{(2)\text{eff}} = \begin{pmatrix} A_{6 \times 6}^{(2)} & B_{6 \times 2}^{(2)} \\ 0_{2 \times 6} & C_{2 \times 2}^{(2)} \end{pmatrix}. \quad (8)$$

The block matrices  $A$  and  $C$  describing the self-mixing of the four-quark operators and the self-mixing of the dipole operators at three loops, respectively, have been calculated in [3]. The block matrix  $B$  describing the mixing of the four-quark operators into the dipole operators at four loops has been determined in [4]. After this calculation the first two steps of the perturbative calculation were completed, that is the effective couplings at the low-energy scale  $\mu \sim m_b$  with resummed logarithms are now known at NNLO accuracy.<sup>3</sup>

In the last step one has to calculate on-shell amplitudes of the operators at the low-energy scale. This is the most difficult part of the NNLO enterprise and it is still under investigation. In order to see what has been done so far, and what still has to be done, we write the decay rate for the partonic decay  $b \rightarrow X_s^{\text{partonic}} \gamma$  as follows,

$$\Gamma^{\text{partonic}}|_{E_\gamma > E_0} = \frac{G_F^2 \alpha_{\text{em}} m_b^5}{32\pi^4} |V_{tb} V_{ts}^*|^2 \sum_{i,j} C_i^{\text{eff}}(\mu) C_j^{\text{eff}}(\mu) G_{ij}(E_0, \mu), \quad (9)$$

where  $G_{ij}(E_0, \mu)$  can again be expanded in terms of  $\tilde{\alpha}_s$ ,

$$G_{ij}(E_0, \mu) = Y_{ij}^{(0)} \delta_{i7} \delta_{j7} + \tilde{\alpha}_s(\mu) Y_{ij}^{(1)}(E_0, \mu) + \tilde{\alpha}_s^2(\mu) Y_{ij}^{(2)}(E_0, \mu) + \dots \quad (10)$$

At NNLO one has to determine the coefficients of  $\tilde{\alpha}_s^2(\mu)$  which, however, has only been done in a complete manner for  $i = j = 7$  [5, 6]. Once we neglect on-shell amplitudes that are proportional to the small Wilson coefficients of the four-quark penguin operators  $O_3$ - $O_6$ , the remaining cases to be considered

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<sup>3</sup> This means large logarithms have been resummed up to  $O(\alpha_s^{n+2} \ln^n(m_b/M_W))$ .

are  $(ij) = (11), (12), (22), (17), (18), (27), (28), (78),$  and  $(88)$ . The large- $\beta_0$  corrections are known in all these cases except for  $(18)$  and  $(28)$  [7, 8, 9]. In addition, effects of the charm and bottom quark masses on the gluon lines are known in all the cases [10, 11]. The other beyond-large- $\beta_0$  corrections have been found only in the limit  $m_c \gg m_b/2$ , except for the  $(78)$  and  $(88)$  cases [12]. This limit has been used to interpolate the unknown beyond-large- $\beta_0$  corrections at  $O(\alpha_s^2)$  to the measured value of  $m_c \approx m_b/4$  [12]. The result for the branching ratio, for  $E_0 = 1.6 \text{ GeV}$ , is given by [13]<sup>4</sup>

$$\text{BR}(\bar{B} \rightarrow X_s \gamma)_{\text{SM}} = (3.15 \pm 0.23) \times 10^{-4} \quad (1\text{S scheme}). \quad (11)$$

The theoretical uncertainty of this NNLO estimate is at the same level as the uncertainty of the current world average reported by HFAG [14],<sup>5</sup>

$$\text{BR}(\bar{B} \rightarrow X_s \gamma)_{\text{exp}} = (3.52 \pm 0.23 \pm 0.09) \times 10^{-4}, \quad (12)$$

which is furthermore expected to come down to the 5% level at the end of the B-factory era.

Here a remark concerning the overall normalization of the theoretical prediction is in order. To reduce parametric uncertainties stemming from the CKM angles as well as from the  $c$ - and  $b$ -quark masses, the partonic decay rate given in (9) is usually normalized using a combination of the  $\bar{B} \rightarrow X_c l \bar{\nu}$  and  $\bar{B} \rightarrow X_u l \bar{\nu}$  decay rates which is reflected by the appearance of the semileptonic phase-space factor

$$C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma(\bar{B} \rightarrow X_c l \bar{\nu})}{\Gamma(\bar{B} \rightarrow X_u l \bar{\nu})} \quad (13)$$

in the analytical expressions [17].<sup>6</sup> Unfortunately, the determination of  $m_c$  and  $C$  from the fit from the measured spectrum of the  $\bar{B} \rightarrow X_c l \bar{\nu}$  decay in the 1S scheme [19] differs from that in the kinetic scheme [20].<sup>7</sup> Using the values for  $m_c$  and  $C$  of the latter determination results in a higher central value for the  $\bar{B} \rightarrow X_s \gamma$  decay rate [21],

$$\text{BR}(\bar{B} \rightarrow X_s \gamma)_{\text{SM}} = (3.25 \pm 0.24) \times 10^{-4} \quad (\text{kinetic scheme}). \quad (14)$$

The difference of  $m_c$  and  $C$  in the 1S and kinetic scheme is likely to be due to different input data, differences in the fit method, and treatment

<sup>4</sup> For a discussion of the residual renormalization scale dependence of the branching ratio at NNLO we refer the reader to [13].

<sup>5</sup> This average includes the measurements from CLEO and BaBar and Belle [15]. The recently published update by Belle [16] has not been taken into account.

<sup>6</sup> The denominator  $\Gamma(b \rightarrow ul\bar{\nu})$  is already known at NNLO accuracy [18].

<sup>7</sup> In [13] the values for  $m_c$  and  $C$  from [19] were adopted.

of theory errors. In this respect, supplementing the fit, for example, by the determination of the  $c$ - and  $b$ -quark masses from sumrules [22] could possibly be helpful to reduce the discrepancy of  $C$  in both schemes.

We should also remark that not all of the aforementioned contributions to the function  $G_{ij}$  entered the analysis of [13]. These are the massive fermionic corrections presented in [6, 10, 11] and the large- $\beta_0$  contributions for  $(ij) \neq (77)$  from [8, 9].<sup>8</sup> These contributions will be included in a future update together with so far unknown contributions, which is, for example, the complete knowledge of  $G_{78}$  [23]. Also the complete calculation of  $G_{27}$  for  $m_c = 0$  is underway [24]. Especially the latter will prove useful in the reduction of the uncertainty stemming from the interpolation in  $m_c$ . Apart from the NNLO corrections also tree-level diagrams with the  $u$ -quark analogues of  $O_{1,2}$  and the four-quark operators  $O_{3-6}$  have been neglected so far [25]. The numerical effect of all of these contributions on the branching ratio is or is expected to remain within the uncertainty of the NNLO estimate given in (11).

Finally, we should mention that there are also cutoff-enhanced corrections which matter close to the endpoint [26]. However, as demonstrated in [21], the resummation of the cutoff-enhanced logarithms overestimates the effect of the  $O(\alpha_s^3)$ -terms for  $E_0 \lesssim 1.6$  GeV. Therefore, the prediction for the branching ratio given in (11) should, at present, be considered as more reliable.

### 3. Non-perturbative corrections

Since the perturbative calculations in  $\bar{B} \rightarrow X_s \gamma$  are now performed at NNLO the non-perturbative corrections become more important. In general well under control are the power-corrections stemming from the OPE of the time ordered products contained in (4). For the self-interference of the electromagnetic operator, that is for  $i=j=7$ , they are known at  $O(\Lambda^2/m_b^2)$  [27] and  $O(\Lambda^3/m_b^3)$  [28]. For all other combinations of  $i$  and  $j$  the time ordered products include contributions in which the photon couples to light quarks, and that causes the breakdown of the OPE. In this case non-perturbative collinear effects [29] as well as power-corrections at  $O(\Lambda^2/m_c^2)$  show up [30]. The combined effect of all of the aforementioned non-perturbative corrections is of around 3% in the branching ratio. Besides, non-perturbative effects appearing at  $O(\alpha_s \Lambda/m_b)$  show up when the photon couples to light quarks. Their size is not known at present and hence a 5% uncertainty related to all the unknown non-perturbative effects has been included in (11). The size of this uncertainty is supported by the

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<sup>8</sup> Also the mixing of the four-quark operators  $O_{1-6}$  into the chromomagnetic dipole operator  $O_8$  [4] was not included in [13].

estimate of the  $O(\alpha_s\Lambda/m_b)$ -corrections in the interference of the electro- and chromomagnetic dipole operators performed in [31]. As pointed out in references [31, 32], the magnitude of the effect considered in [31] could be probed by an improved measurement [33] of the isospin asymmetry

$$\Delta_{0-} = \frac{\Gamma(\bar{B}^0 \rightarrow X_s\gamma) - \Gamma(\bar{B}^- \rightarrow X_s\gamma)}{\Gamma(\bar{B}^0 \rightarrow X_s\gamma) + \Gamma(\bar{B}^- \rightarrow X_s\gamma)}. \quad (15)$$

Finally, we note that a non-perturbative uncertainty appears also when extrapolating the three different measurements performed at CLEO, BaBar and Belle down to the common lower cut  $E_0 = 1.6\text{ GeV}$  in the photon energy. It is accounted for in the error of the world average given in (12).

#### 4. Conclusions

At present the uncertainties in the branching ratio of  $\bar{B} \rightarrow X_s\gamma$  are on the same level on both the theoretical and experimental side. Thanks to the ongoing calculations of the perturbative corrections the uncertainty stemming from this part will further reduce. However, to reach the 5% level or even less on the theoretical side a better understanding of the non-perturbative power-corrections at  $O(\alpha_s\Lambda/m_b)$  is required.

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#### REFERENCES

- [1] M. Misiak and M. Steinhauser, Nucl. Phys. B **683** (2004) 277.
- [2] C. Bobeth, M. Misiak and J. Urban, Nucl. Phys. B **574** (2000) 291.
- [3] M. Gorbahn and U. Haisch, Nucl. Phys. B **713** (2005) 291;  
M. Gorbahn, U. Haisch and M. Misiak, Phys. Rev. Lett. **95** (2005) 102004.
- [4] M. Czakon, U. Haisch and M. Misiak, JHEP **0703** (2007) 008.
- [5] K. Melnikov and A. Mitov, Phys. Lett. B **620** (2005) 69; I. Blokland, A. Czarnecki, M. Misiak, M. Slusarczyk and F. Tkachov, Phys. Rev. D **72** (2005) 033014; H. M. Asatrian, A. Hovhannisyan, V. Poghosyan, T. Ewerth, C. Greub and T. Hurth, Nucl. Phys. B **749** (2006) 325; H. M. Asatrian, T. Ewerth, A. Ferroglia, P. Gambino and C. Greub, Nucl. Phys. B **762** (2007) 212.

- [6] H. M. Asatrian *et al.*, Phys. Lett. B **647** (2007) 173.
- [7] K. Bieri, C. Greub and M. Steinhauser, Phys. Rev. D **67** (2003) 114019.
- [8] Z. Ligeti *et al.*, Phys. Rev. D **60** (1999) 034019.
- [9] A. Ferroglia and U. Haisch, to be published.
- [10] T. Ewerth, Phys. Lett. B **669** (2008) 167.
- [11] R. Boughezal, M. Czakon and T. Schutzmeier, JHEP **0709** (2007) 072.
- [12] M. Misiak and M. Steinhauser, Nucl. Phys. B **764** (2007) 62.
- [13] M. Misiak *et al.*, Phys. Rev. Lett. **98** (2007) 022002.
- [14] E. Barberio *et al.* [Heavy Flavor Averaging Group], arXiv:0808.1297.
- [15] S. Chen *et al.* [CLEO Collab.], Phys. Rev. Lett. **87** (2001) 251807; B. Aubert *et al.* [BaBar Collab.], Phys. Rev. D **77** (2008) 051103; K. Abe *et al.* [Belle Collab.], AIP Conf. Proc. **1078** (2009) 342.
- [16] A. Limosani *et al.* [Belle Collab.], arXiv:0907.1384.
- [17] P. Gambino and M. Misiak, Nucl. Phys. B **611** (2001) 338.
- [18] T. van Ritbergen, Phys. Lett. B **454** (1999) 353; K. Melnikov, Phys. Lett. B **666** (2008) 336; A. Pak and A. Czarnecki, Phys. Rev. Lett. **100** (2008) 241807 and Phys. Rev. D **78** (2008) 114015; M. Dowling, A. Pak and A. Czarnecki, Phys. Rev. D **78** (2008) 074029.
- [19] C. W. Bauer *et al.*, Phys. Rev. D **70**, 094017 (2004);  
A. H. Hoang and A. V. Manohar, Phys. Lett. B **633**, 526 (2006).
- [20] P. Gambino and P. Giordano, Phys. Lett. B **669** (2008) 69.
- [21] M. Misiak, arXiv:0808.3134.
- [22] K. G. Chetyrkin *et al.*, arXiv:0907.2110.
- [23] H. M. Asatrian *et al.*, in preparation.
- [24] T. Schutzmeier, Nucl. Phys. Proc. Suppl. **183** (2008) 215.
- [25] M. Kamiński and M. Misiak, in preparation.
- [26] T. Becher and M. Neubert, Phys. Lett. B **633** (2006) 739, Phys. Lett. B **637** (2006) 251, and Phys. Rev. Lett. **98** (2007) 022003.
- [27] A. F. Falk, M. E. Luke and M. J. Savage, Phys. Rev. D **49** (1994) 3367; I. I. Y. Bigi *et al.*, arXiv:hep-ph/9212227.
- [28] C. W. Bauer, Phys. Rev. D **57** (1998) 5611 [Erratum-ibid. D **60** (1999) 099907]; M. Gremm and A. Kapustin, Phys. Rev. D **55** (1997) 6924.
- [29] A. Kapustin, Z. Ligeti and H. D. Politzer, Phys. Lett. B **357** (1995) 653.
- [30] M. B. Voloshin, Phys. Lett. B **397** (1997) 275; Z. Ligeti, L. Randall and M. B. Wise, Phys. Lett. B **402** (1997) 178; A. K. Grant, A. G. Morgan, S. Nussinov and R. D. Peccei, Phys. Rev. D **56** (1997) 3151; G. Buchalla, G. Isidori and S. J. Rey, Nucl. Phys. B **511** (1998) 594.
- [31] S. J. Lee, M. Neubert and G. Paz, Phys. Rev. D **75** (2007) 114005.
- [32] M. Misiak, talk given at the “Ringberg Workshop on New Physics, Flavors and Jets”, <http://indico.mppmu.mpg.de/indico/conferenceDisplay.py?confid=433>.
- [33] B. Aubert *et al.* [BABAR Collab.], Phys. Rev. D **72** (2005) 052004.